On Uniform Word Problems Involving Bridging Operators on Distributive Lattices

Viorica Sofronie-Stokkermans

Max-Planck-Institut für Informatik, Stuhlsatzenhausweg 85, Saarbrücken, Germany
sofronie@mpi-sb.mpg.de

Abstract. In this paper we analyze some fragments of the universal theory of distributive lattices with many sorted bridging operators. Our interest in such algebras is motivated by the fact that, in description logics, numerical features are often expressed by using maps that associate numerical values to sets (more generally, to lattice elements). We first establish a link between satisfiability of universal sentences with respect to algebraic models and satisfiability with respect to certain classes of relational structures. We use these results for giving a method for translation to clause form of universal sentences, and provide some decidability results based on the use of resolution or hyperresolution. Links between hyperresolution and tableau methods are also discussed, and a tableau procedure for checking satisfiability of formulae of type $t_1 \leq t_2$ is obtained by using a hyperresolution calculus.

1 Introduction

In description logics, numerical information is often associated to concepts. This is achieved, for instance, by using so-called "bridging functions" (terminology introduced in [Ohl01] in the context of set-description languages). An example of a bridging function on a lattice $(L, \cup, \cap, \emptyset, L)$ of sets, where $L \subseteq \mathcal{P}(X)$, is $\mathsf{maxcost} : L \to [0, n]$ defined, for every $A \in L$, by $\mathsf{maxcost}(A) = \mathsf{max}\{\mathsf{cost}(a) \mid a \in A\}$, where $\mathsf{cost} : X \to [0, n]$ is a given map. Then, $\mathsf{maxcost}(\emptyset) = 0$ and, for all $A, B \in L$ $\mathsf{maxcost}(A \cup B) = \mathsf{max}(\mathsf{maxcost}(A), \mathsf{maxcost}(B))$, i.e. $\mathsf{maxcost}$ preserves all finite joins (i.e. it is a *join hemimorphism*). Note that $\mathsf{maxcost}$ does not preserve all meets: in general $\mathsf{maxcost}(A \cap B) \neq \mathsf{min}(\mathsf{maxcost}(A), \mathsf{maxcost}(B)$.

Bridging functions such as $\mathsf{maxcost}$ are special instances of the more general concept of many sorted join hemimorphisms, which can be analyzed in a general algebraic framework. This kind of operators encompass very general types of bridging functions, not necessarily with numerical values.

The main contributions of this paper are the following:

- We formally define a class of many sorted bridging functions between bounded distributive lattices, which we call *many sorted join hemimorphisms.*
- We show that the Priestley representation theorem can be extended in a natural way to encompass such operators.

U. Egly and C.G. Fermüller (Eds.): TABLEAUX 2002, LNAI 2381, pp. 235–250, 2002.

- We show that the results in [SS01] can be adapted also to this more general type of operators:
  - we define a structure-preserving translation to clause form for uniform word problems for such classes of lattices with operators, and
  - obtain resolution-based decision procedures for several classes of algebras of this type.
- We also analyze refinements of resolution such as hyperresolution and its relationship with the definition of tableau calculi (cf. also [HS00]).

In what follows we briefly explain the links between the results presented here and previous work. In [SS99] we gave a method for automated theorem proving in the universal theory of varieties of distributive lattices with operators that were either hemimorphisms or antimorphisms in every argument (hence, generalizations of the modal operators $\diamond$ and $\Box$). In [SS01], the arguments were extended to operators that are hemimorphisms in some arguments and antimorphisms in other arguments. This allowed us to deal in a simple and uniform way with various classes of operators, including both generalizations of modal operators $\Box$ and $\diamond$, but also those obtained by considering e.g. weakened implications, which satisfy identities such as:

$$0 \to z = 1 \ \ (x \vee y) \to z = (x \to z) \wedge (y \to z), \tag{1}$$
$$x \to 1 = 1, \ \ x \to (y \wedge z) = (x \to y) \wedge (x \to z). \tag{2}$$

This allowed us to obtain resolution-based decision procedures for classes of distributive lattices with operators that satisfy *generalized residuation conditions.* The main idea was to think of an operator that is a hemimorphism in some arguments and antimorphism in other arguments as a map of type $\varepsilon_1 \dots \varepsilon_n \to \varepsilon$, where $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, +1\}$, such that $f : L^{\varepsilon_1} \times \dots \times L^{\varepsilon_n} \to L^{\varepsilon}$ is a join hemimorphism, where $L^1 = L$ and $L^{-1} = L^d$, the order-dual of L.

In the present paper we show that the results in [SS99,SS01] can be generalized in a very natural way to many sorted algebras $(\{L_s\}_{s \in S}, \{\sigma_L\}_{\sigma \in \Sigma})$ where, for each sort $s \in S$ L_s is a bounded distributive lattice, endowed with operators which are many sorted join hemimorphisms $f : L_{s_1} \times \dots \times L_{s_n} \to L_s$. Some "bridging functions" ([Ohl01]), such as $\mathsf{maxcost} : L \to [0, n]$, are many sorted join hemimorphisms; others, such as e.g. $\mathsf{mincost}$ are join hemimorphisms from L into the order-dual, $[0, n]^d$, of $[0, n]$.

1.1 Idea

We illustrate the idea of the algorithm we propose on a simple example. Consider the formula ϕ below:

$$(\forall a, b : \mathsf{lat})(\forall c, d : \mathsf{num})\, (\mathsf{maxcost}(a) = c \wedge \mathsf{maxcost}(b) = d \to \mathsf{maxcost}(a \wedge b) \leq c \wedge d)$$

where the variables a and b are of sort lat (range over elements in lattices), the variables c and d are of sort num (range over elements in a numeric domain), and $\mathsf{maxcost}$ is a unary function symbol of type $\mathsf{lat} \to \mathsf{num}$.

Let $D_{01}\mathsf{O}^n$, where n is an arbitrary but fixed natural number, be the class of all algebras with two sorts $S = \{\mathsf{lat}, \mathsf{num}\}$, of the form $(\mathbf{L}, \mathbf{C_n}, \mathsf{maxcost})$, with the property that $\mathbf{L} = (L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, $\mathbf{C_n}$ is the n-element chain with elements $\{1, \dots, n\}$, and $\mathsf{maxcost} : \mathbf{L} \to \mathbf{C_n}$ is a join hemimorphism.

One possibility for proving that ϕ holds in $D_{01}\mathsf{O}^n$ is to show that ϕ is a consequence of the bounded distributive lattice axioms to which a description of the lattice $\{1, \dots, n\}$ is added. Hovever, complications may already arise for one-sorted formulae: as shown in [SS99,SS01], there exist formulae for which even powerful theorem provers such as Spass or Waldmeister could not find reasonably short proofs by using equational resoning in distributive lattices.

Instead, we use the fact that every bounded distributive lattice L is isomorphic to a sublattice of the lattice of all upwards-closed subsets of a preordered set X_L (suprema are unions and infima intersections). In particular, the linearly ordered finite lattice $\{0, \dots, n\}$ is isomorphic to the lattice of all order-filters of $D(n) = (\{\uparrow i \mid 1 \leq i \leq n\}, \subseteq)$, where $\uparrow i = \{j \mid i \leq j \leq n\}$.

As a consequence, $\mathsf{D}_{01}\mathsf{O}^n \models \phi$ iff for every preordered set $(X, \leq)$, ϕ holds for every assignment that replaces its variables of sort lat with upwards-closed subsets of $(X, \leq)$ and those of sort num with upwards-closed subsets of $(D(n), \subseteq)$, if $\vee$ is interpreted as union and $\wedge$ as intersection, and an increasing relation $R_{\mathsf{maxcost}} \subseteq X \times D(n)$ is associated with the map $\mathsf{maxcost}$. We will show that ϕ is true in all algebras in $D_{01}\mathsf{O}^n$ if and only if the following family of set constraints is unsatisfiable:

$$\begin{cases}
(\mathsf{Dom_s}) & (X, \leq) \text{ preordered set} \\
& (D(n), \leq) \text{ is a preordered set with } n \text{ elements, } \uparrow n \leq \cdots \leq \uparrow 1 \\
& (\forall x : \mathsf{lat})(\forall x_i, x_j : \mathsf{num})(x_i \leq x_j \wedge R_{\mathsf{maxcost}}(x, x_i) \to R_{\mathsf{maxcost}}(x, x_j)) \\
(\mathsf{Her_s}) & x_1 \in I_e, x_1 \leq x_2 \to x_2 \in I_e \qquad \text{for all } e \in ST(\phi) \text{ of sort } \mathsf{lat} \text{ or } \mathsf{num} \\
(\mathsf{Ren_s})(\wedge) & I_{a \wedge b} = I_a \cap I_b \qquad I_{c \wedge d} = I_c \cap I_d \\
(\mathsf{m}) & I_{\mathsf{maxcost}(e)} = \{\uparrow i \mid \exists x_1 \in I_e : R_{\mathsf{maxcost}}(x_1, \uparrow i)\} \qquad \text{for } e \in \{a, b, a \wedge b\} \\
(\mathsf{P_s}) & I_{\mathsf{maxcost}(a)} = I_c \\
& I_{\mathsf{maxcost}(b)} = I_d \\
(\mathsf{N_s}) & I_{\mathsf{maxcost}(a \wedge b)} \not\subseteq I_{c \wedge d}
\end{cases}$$

where $ST(\phi)$ is the set of all subterms occurring in ϕ. We encode every set I_e, $e \in ST(\phi)$, by a unary predicate P_e. We obtain again a many-sorted structure with sorts lat and num, where for every $e \in ST(\phi)$, the predicate P_e accepts arguments of the same sort as the expression e.

With this encoding we can reduce the problem of testing the satisfiability of the family of set constraints above to the problem of testing the satisfiability of the following conjunction in first-order logic:

$$\begin{cases}
\text{(Dom)} & & \\
\quad \forall x:\mathsf{s} & x \leq x & \mathsf{s} \in \{\mathsf{lat}, \mathsf{num}\} \\
\quad \forall x,y,z:\mathsf{s} & x \leq y, y \leq z \rightarrow x \leq z & \\
 & \uparrow n \leq \dots \leq \uparrow 1 & \\
\quad \forall x:\mathsf{num}, \forall x_i, x_j:\mathsf{num} & x_i \leq x_j, R_{\mathsf{maxcost}}(x, x_i) \rightarrow R_{\mathsf{maxcost}}(x, x_j) & \\
\text{(Her)} & & \\
\forall x,y:\mathsf{s} & x \leq y, P_e(x) \rightarrow P_e(y) & e \in ST(\phi) \text{ of sort } \mathsf{s} \\
\text{(Ren)} & & \\
\quad (\wedge)\forall x:\mathsf{s} & P_{e_1 \wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x) & e_1 \wedge e_2 \in ST(\phi) \text{ of sort } \mathsf{s} \\
\quad (\vee)\forall x:\mathsf{s} & P_{e_1 \vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x) & e_1 \vee e_2 \in ST(\phi) \text{ of sort } \mathsf{s} \\
\quad (\mathsf{m})\forall x_i:\mathsf{num} & P_{\mathsf{maxcost}(e)}(x_i) \leftrightarrow (\exists x:\mathsf{lat})(P_e(x) \wedge R_{\mathsf{maxcost}}(x, x_i)) & \\
 & & \mathsf{maxcost}(e) \in ST(\phi) \\
\text{(P)} & & \\
\quad \forall x:\mathsf{num} & P_{\mathsf{maxcost}(a)}(x) \leftrightarrow P_c(x) & \\
\quad \forall x:\mathsf{num} & P_{\mathsf{maxcost}(b)}(x) \leftrightarrow P_d(x) & \\
\text{(N)} & & \\
\quad \exists y:\mathsf{num} & P_{\mathsf{maxcost}(a \wedge b)}(y) \wedge \neg P_{c \wedge d}(y). &
\end{cases}$$

We obtain a structure-preserving translation to first-order logic, and, ultimately, to clause form. The satisfiability of the set of clauses obtained this way can be checked for instance by ordered resolution with selection.

In this paper we show that similar ideas can be used for many classes of many sorted bounded distributive lattices with so-called bridging functions. Moreover, we show that refinements of resolution can successfully be used to obtain decision procedures for the universal Horn theories of many such classes, and show how hyperresolution can be used to define a set of sound and complete tableau rules for deciding validity of problems of the type $s \leq t$.

2 Representation of Distributive Lattices with Operators

This section discusses an extension of representation theorems for distributive lattices with bridging operators.

2.1 Distributive Lattices with Bridging Operators

A structure $(L, \vee, \wedge)$, consisting of a non-empty set L together with two binary operations $\vee$ (join) and $\wedge$ (meet) on L, is called *lattice* if $\vee$ and $\wedge$ are associative, commutative and idempotent and satisfy the absorption laws. A *distributive lattice* is a lattice that satisfies either of the distributive laws $(D_\wedge)$ or $(D_\vee)$, which are equivalent in a lattice.

$$(D_\wedge) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (D_\vee) \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice $(L, \vee, \wedge)$ has a *first element* if there is an element $0 \in L$ such that $0 \wedge x = 0$ for every $x \in L$; it has a *last element* if there is an element $1 \in L$ such that $1 \wedge x = x$ for every $x \in L$. A lattice having both a first and a last element is called *bounded*. If $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a bounded lattice we denote by $\mathbf{L}^d$ the order-dual of $\mathbf{L}$, i.e. the lattice $(L, \vee^d, \wedge^d, 0^d, 1^d)$, where for every $x, y \in L$, $x \vee^d y = x \wedge y$, $x \wedge^d y = x \vee y$; $0^d = 1$; and $1^d = 0$. A filter in a lattice $(L, \vee, \wedge)$

is a non-empty order-filter closed under meets. A filter F is said to be *prime* if $F \neq L$ and for every $x, y \in L$, if $x \vee y \in F$ then $x \in F$ or $y \in F$. In what follows the set of prime filters will be denoted by $\mathcal{F}_p(L)$.

Definition 1. *Let $\{\mathbf{L}_s\}_{s \in S}$ be a family of bounded lattices $\mathbf{L}_s = (L_s, \vee, \wedge, 0, 1)$ and let $s_1, \dots, s_n, s \in S$. A join hemimorphism of type $s_1 \dots s_n \to s$ is a function $f : L_{s_1} \times \dots \times L_{s_n} \to L_s$ such that for every $i, 1 \leq i \leq n$,*
(1) $f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = 0$,
(2) $f(a_1, \dots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \dots, a_n) =$
$= f(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \vee f(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n)$.

Example 1.

1. The modal operator $\diamond$ on a Boolean algebra $\mathbf{B}$ is a join hemimorphism. The modal operator $\Box$ on $\mathbf{B}$ is a meet hemimorphism, i.e. a join hemimorphism on the dual $\mathbf{B}^d$ of $\mathbf{B}$. If we consider 2-sorted algebras $(\mathbf{B}, \mathbf{B}^d)$ with sorts $S = \{\mathsf{bool}, \mathsf{bool_d}\}$, $\diamond$ is a join hemimorphism of type $\mathsf{bool} \to \mathsf{bool}$ and $\Box$ is of type $\mathsf{bool_d} \to \mathsf{bool_d}$.
2. Let $\mathbf{L}$ be a lattice, and $(\mathbf{L}, \mathbf{L}^d)$ the 2-sorted algebra with sorts $S = \{\mathsf{lat}, \mathsf{lat_d}\}$. The operation $\to$ satisfying the conditions (1) and (2) on page 236 is a join hemimorphism of type $\mathsf{lat}, \mathsf{lat_d} \to \mathsf{lat_d}$.
3. Let $(\mathbf{L}, \mathbf{C}_{n+1})$ be the 2-sorted algebra with sorts $S = \{\mathsf{lat}, \mathsf{num}\}$, where L is a bounded lattice, and $C_{n+1} = (\{0, 1, \dots, n\}, \vee, \wedge, 0, n)$ is the $n+1$-element chain. A function $f : \mathbf{L} \to \mathbf{C}_{n+1}$ that associates with every element of L an element of $\{0, 1, \dots, n\}$ such that $f(x \vee y) = f(x) \vee f(y)$ and $f(0) = 0$ is a join hemimorphism of type $\mathsf{lat} \to \mathsf{num}$.

2.2 Representation Theorems

We now present a simplified version of Priestley's representation theorem stating that every bounded distributive lattice is isomorphic to a lattice of sets.

Theorem 1 ([Pri70]). *Let $\mathbf{L}$ be a distributive lattice, let $D(\mathbf{L}) = (\mathcal{F}_p(\mathbf{L}), \subseteq)$ be the partially-ordered set having as points the prime filters of $\mathbf{L}$, ordered by inclusion, and let $\mathcal{H}(D(\mathbf{L}))$ be the lattice of all upwards-closed subsets of $D(\mathbf{L})$. Then the map $\eta_L : L \to \mathcal{H}(D(\mathbf{L}))$, defined for every $x \in L$ by $\eta_L(x) = \{F \in \mathcal{F}_p(\mathbf{L}) \mid x \in F\}$ is an injective lattice homomorphism.*

In what follows we will refer to the space $D(\mathbf{L})$ as the dual of $\mathbf{L}$.

It was shown that operators on a bounded distributive lattice induce in a canonical way maps resp. relations on $D(\mathbf{L})$ [JT52,Gol89,SS00,SS01]. We now show that these canonical definitions can be formulated in a very general way, which enables us to extend them to many sorted join hemimorphisms. If $f : \mathbf{L}_{s_1} \times \dots \times \mathbf{L}_{s_n} \to \mathbf{L}_s$ is a join hemimorphism, then a relation $R_f \subseteq D(\mathbf{L}_{s_1}) \times \dots \times D(\mathbf{L}_{s_n}) \times D(\mathbf{L}_s)$ can be defined by:

$$R_f(F_1, \dots, F_n, F) \text{ iff } f(F_1, \dots, F_n) \subseteq F.$$

Proposition 1. *Let $\{\mathbf{L}_s\}_{s \in S}$ be a family of bounded distributive lattices. Let $f : \mathbf{L}_{s_1} \times \cdots \times \mathbf{L}_{s_n} \to \mathbf{L}_s$ be a join hemimorphism of type $s_1 \dots s_n \to s$. Then R_f is an increasing relation*[1]*.*

Proposition 1 justifies the definition of S-sorted *RT Σ-relational structures.*

Definition 2. *An* S-sorted *RT* Σ-relational structure $(\{(X_s, \leq)\}_{s \in S}, \{R_X\}_{R \in \Sigma})$ *is an S-sorted family of sets, each endowed with a reflexive and transitive relation $\leq$ and with additional maps and relations indexed by Σ, where, if $R \in \Sigma$ is of type $s_1 \dots s_n \to s$, $R_X \subseteq X_{s_1} \times \cdots \times X_{s_n} \times X_s$ is increasing.*

For every S-sorted *RT* Σ-relational structure $\mathbf{X} = (\{(X_s, \leq)\}_{s \in S}, \{R_X\}_{R \in \Sigma})$, we denote by $\mathcal{H}(\mathbf{X})$ the many sorted algebra $(\{(\mathcal{H}(X_s), \cup, \cap, \emptyset, X_s)\}_{s \in S}, \{f_R\}_{R \in \Sigma})$, where, for every $s \in S$, $(\mathcal{H}(X_s), \cup, \cap, \emptyset, X_s)$ is the bounded distributive lattice of all hereditary (i.e. upwards-closed with respect to $\leq$) subsets of X_s, and the operators $\{f_R\}_{R \in \Sigma}$ are defined as follows.

If $R \subseteq X_{s_1} \times \cdots \times X_{s_n} \times X_s$ is an increasing relation then $f_R : \mathcal{H}(X_{s_1}) \times \cdots \times \mathcal{H}(X_{s_n}) \to \mathcal{H}(X_s)$ is defined, for every $(U_1, \dots U_n) \in \mathcal{H}(X_{s_1}) \times \cdots \times \mathcal{H}(X_{s_n})$ by

$$f_R(U_1, \dots, U_n) = R^{-1}(U_1, \dots, U_n),$$

where $R^{-1}(U_1, \dots, U_n) = \{x \mid \exists x_1 \dots x_n (x_1 \in U_1, \dots, x_n \in U_n, R(x_1, \dots, x_n, x))\}$.

Proposition 2. *Let X be an S-sorted RT Σ-relational structure. If $R \in \Sigma$ is of type $s_1 \dots s_n \to s$ then $f_R : \prod_{i=1}^n \mathcal{H}(X_{s_i}) \to \mathcal{H}(X_s)$ is a join hemimorphism.*

We will denote the class of all S-sorted distributive lattices with operators in Σ by DLO^S_Σ. The class of S-sorted *RT* Σ-relational structures will be denoted by RT^S_Σ. In the one-sorted case the index S will usually be omitted. The following result extends the representation theorems in [Gol89,SS00,SS02] to the more general classes of operators we consider here.

Theorem 2. *For every $\mathbf{A} = (\{\mathbf{L}_s\}_{s \in S}, \{f_A\}_{f \in \Sigma}) \in \mathsf{DLO}^S_\Sigma$, $D(\mathbf{A}) \in RT^S_\Sigma$, and $\eta_A : A \to \mathcal{H}(D(A))$ defined for every $s \in S$ and every $x \in L_s$ by $\eta^s_A(x) = \{F \in \mathcal{F}_p(\mathbf{L}_s) \mid x \in F\}$ is an injective homomorphism between algebras in DLO^S_Σ.*

Proof: Similar to the proof of Theorem 14 in [SS02]. □

3 The Universal Theory of DLO^S_Σ and Subclasses Thereof

In this section we show that the representation theorems discussed before allow, under certain conditions, to avoid the explicit use of the full algebraic structure of distributive lattices with S-sorted bridging operators and use, instead, lattices of sets over structures in RT^S_Σ. This justifies a structure-preserving translation to clause form. The proofs, which we do not provide here, are easy generalizations of those in [SS01,SS02].

[1] A relation $R \subseteq X_1 \times \cdots \times X_n \times X$ is increasing if for every $\overline{x} \in X_1 \times \cdots \times X_n$, and every $y, y' \in X$ (if $R(\overline{x}, y)$ and $y \leq y'$ then $R(\overline{x}, y')$).

3.1 Generalities

Let $\mathcal{V}$ be a class of (many sorted) algebras. The *universal theory* of $\mathcal{V}$ is the collection of those closed formulae valid in $\mathcal{V}$ which are of the form

$$\forall x_1 \dots \forall x_k \left(\bigwedge_{i=1}^{m} ((\neg)t_{i1} = s_{i1} \vee \dots \vee (\neg)t_{in_i} = s_{in_i}) \right). \tag{3}$$

Since a conjunction is valid iff all its conjuncts are valid, we can restrict, without loss of generality, to formulae of the type $(\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2})$.

The *universal Horn theory* of $\mathcal{V}$ is the collection of those closed formulae valid in $\mathcal{V}$ which are of the form $\forall x_1 \dots \forall x_n (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow t_{j1} = t_{j2})$.

If $\mathcal{V}$ is a class of algebras which is closed under direct products then, by a result of McKinsey, the decidability of the universal Horn theory of $\mathcal{V}$ implies the decidability of the universal theory of $\mathcal{V}$.

3.2 A Link between Algebraic and Relational Models

We establish a link between truth of universal sentences in classes of distributive lattices with operators and truth in classes of S-sorted RT Σ-relational structures. We consider subclasses $\mathcal{V}$ of DLO^S_Σ that satisfy the following condition:

(K) There exists a $\mathcal{K} \subseteq RT^S_\Sigma$ such that (i) for every $\mathbf{A} \in \mathcal{V}$, $D(\mathbf{A}) \in \mathcal{K}$;
(ii) for every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \in \mathcal{V}$.

Proposition 3 ([SS01]). *Assume that $\mathcal{V}$ satisfies condition (K). Then for every $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2})$,*

$$\mathcal{V} \models \phi \textit{ if and only if for every } \mathbf{X} \in \mathcal{K}, \mathcal{H}(\mathbf{X}) \models \phi.$$

For automated theorem proving it is important to find subclasses of RT^S_Σ with good theoretical and logic properties, for instance subclasses which are *first-order definable*. Although this is not always possible, such classes can often be obtained by abstracting properties of the Priestley duals of algebras in $\mathcal{V}$.

Lemma 1. *Condition (K) holds in the following cases:*

1. $\mathcal{V} = \mathsf{DLO}^S_\Sigma = \{(\{\mathbf{L}_s\}_{s \in S}, \{f\}_{f \in \Sigma}) \mid \mathbf{L}_s \in \mathsf{D}_{01}$ *for all* $s \in S; f : \prod_{i=1}^{n} \mathbf{L}_{s_i} \rightarrow \mathbf{L}_s$ *join hemimorphism, for every* $f \in \Sigma_{s_1 \dots s_n \rightarrow s}\}$ *and* $\mathcal{K} = RT^S_\Sigma$.
2. $\mathcal{V} = \mathsf{BAO}^S_\Sigma = \{(\{\mathbf{B}_s\}_{s \in S}, \{f\}_{f \in \Sigma}) \mid \mathbf{B}_s \in \mathsf{Bool}$ *for all* $s \in S; f : \prod_{i=1}^{n} \mathbf{B}_{s_i} \rightarrow \mathbf{B}_s$ *join hemimorphism, for every* $f \in \Sigma_{s_1 \dots s_n \rightarrow s}\}$ *and* $\mathcal{K} = R^S_\Sigma$ *the subclass of* RT^S_Σ *consisting only of those S-sorted spaces in which all supports are discretely ordered.*
3. *If* $\mathbf{A} \in \mathsf{D}_{01}$ *is an arbitrary but fixed* finite *lattice and* $S = \{\mathsf{lat}, \mathsf{num}\}$*:* $\mathcal{V} = \mathsf{DLO}^A_\Sigma = \{(\mathbf{L}, \mathbf{A}, \{f_L\}_{f \in \Sigma_L}, \{f_b\}_{f \in \Sigma_b}) \mid \mathbf{L} \in \mathsf{D}_{01};\ f_L : \mathbf{L}^k \rightarrow \mathbf{L}$ *join hemimorphism, for every* $f \in \Sigma_L,$ *of type* $\mathsf{lat}^k \rightarrow \mathsf{lat}; f_b : \mathbf{L}^m \rightarrow \mathbf{A}$ *join hemimorphism for every* $f \in \Sigma_b,$ *of type* $\mathsf{lat}^m \rightarrow \mathsf{num}\}$, *and* $\mathcal{K} = \{(X, D(\mathbf{A}), \{R_f\}_{f \in \Sigma_L}, \{R_g\}_{g \in \Sigma_b}) \mid (X, \{R_f\}_{f \in \Sigma_L}) \in RT_{\Sigma_L}$ *and* $R_g \subseteq X^m \times D(\mathbf{A})$ *increasing for all* $g \in \Sigma_b$ *of type* $\mathsf{lat}^m \rightarrow \mathsf{num}\}$.

3.3 Structure-Preserving Translation to Clause Form

We show that, if a subclass $\mathcal{V}$ of $\mathsf{DLO}^{\mathsf{S}}_{\Sigma}$ satisfies condition (K) for some first-order definable subclass $\mathcal{K}$ of RT^S_{Σ}, then the problem of checking whether a formula $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$ holds in $\mathcal{V}$ can be reduced to the problem of checking the satisfiability of a set of clauses.

Let $ST(\phi)$ be the set of all subterms of s_{il} and t_{jp}, $1 \leq i \leq n, 1 \leq j \leq m, l, p \in \{1, 2\}$ (including the variables and s_{il}, t_{jp} themselves).

Proposition 4. *Let $\mathcal{K} \subseteq RT^S_{\Sigma}$. The following are equivalent:*

(1) *For every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$.*
(2) *For every $\mathbf{X} = (\{(X_s, \leq)\}_{s \in S}, \{R\}_{R \in \Sigma}) \in RT^S_{\Sigma}$ and every family of subsets of X indexed by all subterms of ϕ, $\{I_e \subseteq X_s \mid e \in ST(\phi)$ of sort $s \in S\}$, if:*

$$\begin{cases}
(\mathsf{Dom}_s)\ \mathbf{X} \in \mathcal{K}, & \\
(\mathsf{Her}_s)\ I_e \in \mathcal{H}(X_s) & \forall e \in ST(\phi) \text{ of sort } s,\\
(\mathsf{Ren}_s)\ (1,0) \quad I_{1_s} = X_s, \quad I_{0_s} = \emptyset, & \\
\quad (\wedge) \quad I_{e_1 \wedge e_2} = I_{e_1} \cap I_{e_2}, & \\
\quad (\vee) \quad I_{e_1 \vee e_2} = I_{e_1} \cup I_{e_2}, & \\
\quad (\Sigma_{s_1 \dots s_n \rightarrow s}) \quad I_{f(e_1, \dots, e_n)} = R_f^{-1}(I_{e_1}, \dots, I_{e_n}), & \\
(\mathsf{P}_s) \quad I_{s_{i1}} = I_{s_{i2}} & \text{for all } i = 1, \dots, n,
\end{cases}$$

then

(C_s) *for some $j \in \{1, \dots, m\}$ $I_{t_{j1}} = I_{t_{j2}}$,*

where the rules in (Σ) range over all terms in $ST(\phi)$ starting with an operator in $\Sigma_{s_1 \dots s_n \rightarrow s}$. (We used the abbreviation $R^{-1}(U_1, \dots, U_n) := \{x \mid \exists x_1 \in U_1 \dots \exists x_n \in U_n : R(x_1, \dots, x_n, x)\}$.)

If the class $\mathcal{K}$ is first-order definable, Proposition 4 justifies a structure-preserving translation of universal formulae to sets of clauses.

Proposition 5. *Let $\mathcal{K}$ be a subclass of RT^S_{Σ} which is definable by a finite set C of first-order sentences. Then the following are equivalent:*

(1) For every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$.
(2) The conjunction of $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \dots \cup (\mathsf{N_m})$ *is unsatisfiable, where:*

$$\begin{array}{llrcll}
(\mathsf{Dom}) & & C, & & & \\
& & \leq \subseteq X_s \times X_s & \text{is} & \text{reflexive and transitive} & \text{for every sort } s \in S,\\
& & R_f \subseteq \prod_{i=1}^{n+1} X_{s_i} & \text{is} & \text{increasing} & \text{for every } f \in \Sigma_{s_1 \dots s_n \rightarrow s_{n+1}},\\
(\mathsf{Her}) & \forall x, y & (x \leq y \wedge P_e(x) & \rightarrow & P_e(y)) & \\
(\mathsf{Ren}) & & & & & \\
\quad (1) & \forall x & P_{1_s}(x) & & & \text{for every sort } s \in S,\\
\quad (0) & \forall x & \neg P_{0_s}(x) & & & \text{for every sort } s \in S,\\
\quad (\wedge) & \forall x & (P_{e_1 \wedge e_2}(x) & \leftrightarrow & P_{e_1}(x) \wedge P_{e_2}(x)) & \\
\quad (\vee) & \forall x & (P_{e_1 \vee e_2}(x) & \leftrightarrow & P_{e_1}(x) \vee P_{e_2}(x)) & \\
\quad (\Sigma) & \forall x & (P_{f(e_1, \dots, e_n)}(x) & \leftrightarrow & \exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i) \wedge R_f(x_1, \dots, x_n, x))) & \\
(\mathsf{P}) & \forall x & (\bigwedge_{i=1}^n P_{s_{i1}}(x) & \leftrightarrow & P_{s_{i2}}(x)) & \\
(\mathsf{N}_1) & \exists x_1 & (P_{t_{11}}(x_1) & \not\leftrightarrow & P_{t_{12}}(x_1)) & \\
& \dots & & \dots & & \\
(\mathsf{N}_m) & \exists x_m & (P_{t_{m1}}(x_m) & \not\leftrightarrow & P_{t_{m2}}(x_m)) &
\end{array}$$

where the unary predicates P_e are indexed by elements in $ST(\phi)$, and the formulae in Σ range over all operators $f \in \Sigma_{s_1 \dots s_n \to s}$.

In addition, polarity of subformulae can be used for using only one direction of the implications in (Ren). Similar ideas can be used for obtaining translations to clause form for formulae of the form $\bigwedge_{i=1}^n s_{i1} \leq s_{i2} \to \bigwedge_{j=1}^m t_{j1} \leq t_{j2}$. Then only the direct implications are necessary in (P) and (N).

Theorem 3. *Assume that $\mathcal{V}$ and $\mathcal{K}$ satisfy condition (K), where $\mathcal{K}$ is a class of RT Σ-structures definable by a finite set C of first-order sentences. The following are equivalent:*
(1) $\mathcal{V} \models \phi$.
(2) *The conjunction of* $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \dots \cup (\mathsf{N_m})$ *(as defined above) is unsatisfiable.*

Proof: Direct consequence of Propositions 3, 4 and 5. □

4 Some Decidability Results

In the following sections we present some examples in which decidability results can be obtained easily. We show that

- orderer resolution with selection decides in exponential time the universal Horn theory of DLO^S_Σ and of DLO^A_Σ, where $A \in \mathsf{D}_{01}$ is finite;
- hyperresolution is a decision procedure for deciding whether $t_1 \leq t_2$ holds in the class DLO^S_Σ and in the class DLO^A_Σ, where $A \in \mathsf{D}_{01}$ is finite;
- hyperresolution can be used to synthesize tableau calculi.

4.1 Ordered Resolution with Selection

Let $\succ$ be a total well-founded ordering on ground atoms, and let S be an arbitrary selection function that assigns with every clause a multiset of negative selected literals. Let $R^{\succ}_S$ be the following inference system for ground clauses, consisting of ordered resolution with selection S and ordered factoring:

Ordered resolution:

$$\frac{C \vee A \quad D \vee \neg A}{C \vee D}$$

where (i) A is strictly maximal[2] in $C \vee A$, and C contains no selected atoms; (ii) $\neg A$ is either selected by S in $D \vee \neg A$ or else $D \vee \neg A$ contains no selected literals and $\neg A$ is maximal in $D \vee \neg A$.

Ordered (positive) factoring:

$$\frac{C \vee A \vee A}{C \vee A}$$

where A is a positive atom which is maximal in C, and no atom in C is selected.

[2] We say that a literal L is maximal in a clause C if $L' \succ L$ for no literal L' in C; and that L is strictly maximal in C if $L' \succeq L$ for no $L' \neq L$ in C.

Ordered resolution with selection can be lifted to non-ground clauses by viewing non-ground expressions to represent the set of their ground instances and by employing unification to avoid the explicit enumeration of ground instances (for details cf. e.g. [BG01]).

The results of [SS01] can be easily adapted to prove the following theorem.

Theorem 4. *Ordered resolution with selection decides in time* exponential in the size of the input *if the arity of operators in Σ has an upper bound, and* exponential in the square of the size of the input *in general the universal Horn theory of (1)* DLO^S_Σ*, and (2)* DLO^A_Σ*, where A is a finite distributive lattice.*

Idea of the Proof: (1) The results of [SS01], Section 5.1 can be easily adapted to prove (1). As pointed out in [SS01], the selection strategy we adopt for this purpose shows, as a by-product, that in this case inferences with the clauses containing the $\leq$ symbol are not needed for refutational completeness.
(2) In a similar way we can show that inferences with the clauses containing the $\leq$ symbol applied to arguments of sort lat are not needed in the case of DLO^A_Σ. Since $D(A)$ is finite, the monotonicity and heredity rules for sort num, can be replaced with their instances with elements in $D(A)$. For instance the monotonicity and heredity rule can alternatively be expressed by:

$$R_f(x_1, \dots, x_n, a) \rightarrow R_f(x_1, \dots, x_n, b) \qquad \text{for all } a, b \in D(A), a \leq b \qquad (4)$$

$$P_e(a) \rightarrow P_e(b) \qquad \text{for all } a, b \in D(A), a \leq b \qquad (5)$$

We can now introduce $D(A)$ copies for every predicate symbol with last argument of sort num, e.g. by replacing, for every $a \in D(A)$, $R_f(x_1, \dots, x_n, a)$ with $R^a_f(x_1, \dots, x_n)$ and $P_e(a)$ with P^a_e. Arguments in [SS01], Section 5.2 can now be applied and also in this case yield the desired complexity results. □

Similar arguments can be also used for (many sorted) Boolean algebras with operators, by considering, in addition, the renaming rules for Boolean negation.

4.2 Hyperresolution

Hyperresolution can be simulated by resolution with maximal selection. This means that the selection function selects all the negative literals in any non-positive clause. Let H be the calculus consisting of negative hyperresolution, (positive) factoring, splitting and tautology deletion.

Negative hyperresolution:

$$\frac{C_1 \vee A_1 \quad \dots \quad C_n \vee A_n \qquad \neg A_1 \vee \dots \vee \neg A_n \vee D}{C_1 \vee \dots \vee C_n \vee D}$$

where D and $C_i \vee A_i$, $1 \leq i \leq n$ are positive clauses; and no A_i occurs in C_i.

Hyperresolution can be combined with ordering restrictions, and can be lifted to non-ground clauses by viewing non-ground expressions to represent the set of their ground instances and by employing unification to avoid the explicit enumeration of ground instances (cf. e.g. [BG01]). Ordered resolution and hyperresolution can be combined with splitting: Suppose that a set N of clauses contains a clause $C = C_1 \vee C_2$, where C_1 and C_2 are non-trivial and have no variables in common. In order to show that N is unsatisfiable, one proves that $(N\backslash\{C\}) \cup \{C_i\}$, $i = 1, 2$ are both unsatisfiable. The components in the variable partition of a clause are called split components. Two split components do not share variables. A clause that cannot be split is called *a maximally split clause.*

We now show that hyperresolution with splitting can be used for the simpler problem of checking whether $\mathsf{DLO}^S_\Sigma \models t_1 \leq t_2$ or $\mathsf{DLO}^A_\Sigma \models t_1 \leq t_2$. Though much more special than uniform word problems, problems of this type often occur in non-classical logics. For instance, in some relevant logics [AB75] or in variants of the (full) Lambek calculus [Ono93], it can be proved that a formula ϕ is a theorem iff $\mathcal{V} \models \phi \geq e$, where $\mathcal{V}$ is the class of all algebraic models of the respective logic and e is a special constant.

Theorem 5. *For all terms t_1, t_2, H decides whether $\mathsf{DLO}^S_\Sigma \models t_1 \leq t_2$.*

Proof: By Theorem 3 and Lemma 1 as well as the fact that, by the proof of Theorem 4, in this case all clauses containing the symbol $\leq$ can be ignored, $\mathsf{DLO}^S_\Sigma \models t_1 \leq t_2$ iff the clause form of $(\mathsf{Ren}) \cup (\mathsf{N})$ is unsatisfiable. Since the premise of the formula ϕ is empty, there are no clauses in (P). (N) consists of the unit clauses $P_{t_1}(c)$ and $\neg P_{t_2}(c)$ for some constant c.

We show that any H-derivation terminates on the set of clauses $(\mathsf{Ren}) \cup (\mathsf{N})$ associated with $\phi = (t_1 \leq t_2)$, in which all predicates P_e are fully labeled, in the sense that for every non-variable subterm $e \in ST(\phi)$, the precise occurrence π of e in t_1 or t_2 is indicated (e.g. in the form $P_e^{t_i^\pi}$).

All non-unit clauses of (Ren) contain a negative (hence selected) literal. Therefore they can only be used as negative premises of resolution steps. $(\mathsf{Ren})(1)$ can only be used in inferences with the clause $\neg P_{t_2}(c)$ if $t_2 = 1$, and $(\mathsf{Ren})(0)$ only in inferences with $P_{t_1}(c)$, if $t_1 = 0$; in both cases the empty clause is obtained.

Except for $(\mathsf{Ren})(1)$, at the beginning there is only one candidate for a positive premise, namely the positive (ground) conjunct of (N), $P_{t_1}(c)$. It can be checked that hyperresolution inferences with such unit ground clauses will, in a first step, generate maximally split clauses of the form $P_e(s)$ or $R_f(c_1(s), \dots, c_n(s), s)$ for some term s which contains only the constant c, and such that (i) the terms e are subterms of t_1, (ii) the Skolem functions introduced by $(\mathsf{Ren})(\Sigma)$ that occur in s are all labeled with subterms of t_1, (iii) for every literal $P_e(s)$ obtained this way, the sum between the height of e and the height of s does not exceed the height $h(t_1)$ of t_1, and (iv) for every literal $R_f(c_1^{f(e_1, \cdots e_n)}(s), \dots, c_n^{f(e_1, \cdots e_n)}(s), s)$ obtained this way, such that $c_1^{f(e_1, \cdots e_n)}, \dots, c_n^{f(e_1, \cdots e_n)}$ are Skolem functions introduced by $(\mathsf{Ren})(\Sigma)$, the sum between the height of e and the height of s is does not exceed $h(t_1) - 1$. Since the depth of the arguments can be bounded, this part of the procedure obviously terminates. Moreover, it can be seen that for each argument s generated this way, all labels of the Skolem functions occurring in s correspond to subterms occurring along one branch in the tree representation of the term t_1. This shows that the number of arguments of literals generated this way is bounded by the number of subterms of t_1. Hence, the number of positive ground atoms generated in this part of the procedure is polynomial in the size of t_1. Note that in this phase only inferences with clause forms of direct implications in Ren are possible.

After generating all unit clauses of the form $P_p(s)$, where p is a propositional variable (by the remarks above, in all such cases the height of s does not exceed $h(t_1)$), inferences with rules in (Ren) involving subformulae of t_2 are possible. Only inferences with clause forms of inverse implications in Ren lead to non-redundant clauses. A similar argument as before shows that also in this case the depth of the arguments can grow with at most $h(t_2)$. This shows that H terminates on the set of clauses associated with $\phi = t_1 \leq t_2$. □

The termination proof above shows that the number of different literals in any derivation tree is polynomial in the size of the input. The arguments in Theorem 5 can be also adapted to many sorted Boolean algebras with operators for which every term has a negation normal form (in particular, for modal algebras).

Theorem 6. *Let A be an arbitrary, but fixed, finite bounded distributive lattice. Then for all terms t_1, t_2, H decides whether $\mathsf{DLO}^A_\Sigma \models t_1 \leq t_2$.*

Idea of the proof: Without loss of generality we assume that t_1 and t_2 are formulae of sort A (otherwise, as all bridging functions are of type $\mathsf{lat}\ldots\mathsf{lat} \rightarrow \mathsf{num}$ no subterms of sort num occur in $ST(t_1 \leq t_2)$, and so Theorem 5 can be applied.). The proof proceeds along the same lines as that of Theorem 5, with the following differences. Instead of using a Skolem constant c for the negation of the premise, we test unsatisfiability of (Dom) ∪ (Her) ∪ (Ren) ∪ ($\mathsf{N_a}$) for all variants of (N), ($\mathsf{N_a}$) : $P_{t_1}(a) \wedge \neg P_{t_2}(a)$, where $a \in D(A)$. Heredity clauses for A and monotonicity conditions of the form (5) resp. (4) for bridging functions and predicates of sort num have to be also taken into account. The inferences of $P_{t_1}(a)$ with the heredity clauses for P_{t_1} yield all unit ground literals of the form $P_{t_1}(b)$, $a \leq b \in D(A)$. The proof continues along the same lines as that of Theorem 5. □

4.3 Tableau Calculi

Selection refinements of resolution, and in particular hyperresolution, are closely related to standard (modal) tableau calculi [HS99,HS00].

A tableau is a finitely branching tree whose nodes are sets of labeled formulae. Tableaux are used for testing satisfiability of formulae. If ϕ is a formula to be tested for satisfiability, the root node is the set $\{a : \phi\}$. Successor nodes are constructed according to a set (Exp) of expansion rules of the form

$$\frac{X}{X_1 \mid \cdots \mid X_n}$$

The expansion rule above can be applied for a formula F if F is an instance of X. n successor nodes are created which contain the formulae of the current node and the appropriate instances of X_i. A branch in a semantic tableau is *closed* if it contains $\perp$ or labeled formulae of the form $a : F$ and $a : \neg F$. Otherwise the branch is called *open*. A tableau is *closed* if each of its branches is closed. A formula ϕ is satisfiable (w.r.t. (Exp)) if a tableau can be constructed (with the set (Exp) of expansion rules) which contains an open maximal branch.

In many papers in which tableau methods are given for modal or description logics, the formula ϕ whose satisfiability is tested is supposed to be in negation normal form. This ensures that all subformulae of ϕ that are not propositional variables have positive polarity, hence only the direct implications of the renaming rules need to be used in a hyperresolution procedure.

When checking satisfiability of formulae of type $t_1 \not\leq t_2$, t_1 (or, in clause form, P_{t_1}) has positive polarity and t_2 (in clause form, P_{t_2}) has negative polarity. In what follows we show that tableau rules as used in modal and description logics can be formulated when t_2 is a constant k. In this case, the root node is the set $\{a : t_1, a : \neg k\}$. Successor nodes are constructed according to the set T_Σ^S of expansion rules below. We will also indicate how a (non-standard) variant of tableaux with polarities can be used for checking satisfiability of formulae of type $t_1 \not\leq t_2$. In that case the root node is the set $\{a : t_1^p, a : \neg t_2^n\}$, and successor nodes are constructed according to a set $T_\Sigma^S(ext)$ of rules.

Let T_Σ^S be the following set of tableau rules:

$$(\perp)\ \frac{s:0}{\perp} \qquad \frac{s:\neg 1}{\perp} \qquad \frac{s:e, s:\neg e}{\perp} \qquad (\wedge)\ \frac{s:e_1\wedge e_2}{s:e_1, s:e_2} \qquad (\vee)\ \frac{s:e_1\vee e_2}{s:e_1 \mid s:e_2}$$

$$(f)\ \frac{s : f(e_1, \dots, e_n)}{(s_1, \dots, s_n, s) : R_f, s_1 : e_1, \dots, s_n : e_n} \quad \text{with } s_1, \cdots s_n \quad \text{new to the branch.}$$

Theorem 7. *The formula $t_1 \not\leq k$, where k is a constant, is unsatisfiable in DLO_Σ^S iff a tableau in which every branch is closed can be constructed from $\{c{:}t_1, c{:}\neg k\}$ using the set T_Σ^S of tableau rules (and, in addition, $\frac{s:\neg k}{s:\neg R_k}$ or $\frac{s:R_k}{s:k}$).*

Proof: The proof uses ideas on the link between resolution and tableaux in [HS99, HS00]. By the soundness, completeness and termination of the hyperresolution calculus H, $t_1 \not\leq k$ is unsatisfiable in DLO_Σ^S iff on all split branches the empty clause is derived from (Ren) and (N) in H.

Assume that $t_1 \not\leq k$ is unsatisfiable in DLO_Σ^S. Then the empty clause is derived from (Ren) and (N) in H on all branches caused by splitting. By polarity considerations, only the direct implications of the definitions of subterms of t_1 in (Ren) are used. If $k = 1$, $\neg P_k(c)$ produces the empty clause with $P_1(x)$. If $k = 0$, $\neg P_k(c)$ is subsumed by $\mathsf{Ren}(0)$. If $k \not\in \{0, 1\}$ the inference of $\neg P_k(c)$ with (Ren)(k) produces $\neg R_k(c)$. Since tableau rules are macro-inference steps of H on the clause form of the direct implications of the definitions in (Ren), based on the hyperresolution proof of the empty clause, a tableau can be constructed from $\{c : t_1, c : \neg k\}$ in which every branch is closed.

Conversely, assume that a tableau can be constructed from $\{c : t_1, c : \neg k\}$ in which every branch is closed. Let h be the map that associates literals to labeled formulae defined by $h(s : e) = h_1(e)(h_2(s))$, where $h_1(e) = P_e$ for every subformula e of $t_1 \leq t_2$, $h_1(R_f) = R_f$; $h_2(s_i) = c_i^{f(e_1,\dots,e_m)}(h(s))$ if s_i was introduced by (f), where $c_i^{f(e_1,\dots,e_m)}$ is the Skolem function associated with e_i and $f(e_1, \cdots e_n)$; and $h_2(s) = s$ otherwise. Then derivations in H correspond to the tableau rules above. For instance, the following derivation corresponds to rule (f). From $P_{f(e_1,\dots,e_n)}(s)$ derive $P_{e_i}(c_i^{f(e_1,\dots,e_n)})(s)$, $i = 1, \dots, n$ and $R_f(c_1^{f(e_1,\dots,e_n)}(s), \dots, c_n^{f(e_1,\dots,e_n)}(s), s)$ using $(\mathsf{Ren})(\Sigma)$ in $n+1$ steps. □

A set of tableau rules for checking satisfiability in DLO_Σ^A of formulae of the form $t_1 \leq k$ can be obtained from T_Σ^S by adding the rules:

$$(\mathsf{Her_A})\ \frac{a : e}{\bigwedge_{b \in D(A), b \geq a} b : e} \qquad (\mathsf{Mon_g})\ \frac{(s_1, \dots, s_n, a) : R_g}{\bigwedge_{b \in D(A), b \geq a} (s_1, \dots, s_n, b) : R_g}$$

for all elements $a \in D(A)$, where the labels $s_1, \dots s_n$ are of type lat, e is a formula of type A and g a bridging function with values of type A.

Validity of formulae of the form $t_1 \leq t_2$, where t_2 is a term can be tested by using a fairly unusual extension of the notion of tableaux to what we call *tableaux with polarities*, in which the direction in which a rule is applied is determined by the polarity of the formula. Both polarities are associated with propositional variables. Let $T_\Sigma^S(ext)$ be the set of rules containing $(\bot)$ and:

$$(\wedge^p)\ \frac{s : (e_1 \wedge e_2)^p}{s : e_1^p, s : e_2^p} \quad (\vee^p)\ \frac{s : (e_1 \vee e_2)^p}{s : e_1^p \mid s : e_2^p} \quad (\wedge^n)\ \frac{s : e_1^n, s : e_2^n}{s : (e_1 \wedge e_2)^n} \quad (\vee^n)\ \frac{s : e_i^n}{s : (e_1 \vee e_2)^n}$$

$$(f^p)\ \frac{s : f(e_1, \dots, e_n)^p}{(s_1, \dots, s_n, s) : R_f, s_1 : e_1^p, \dots, s_n : e_n^p} \quad (f^n)\ \frac{(s_1, \dots, s_n, s) : R_f, s_1 : e_1^n, \dots, s_n : e_n^n}{s : f(e_1, \dots, e_n)^n}$$

(with $s_1, \cdots s_n$ new to the branch)

These rules encode macro-inference steps of H with the clause form of direct implications in (Ren) for subterms of t_1 $((\wedge^p), (\vee^p), (f^p))$ resp. the inverse implications in (Ren) for subterms of t_2 $((\wedge^n), (\vee^n), (f^n))$. Similar arguments as those used in Theorem 7 can be used to show that the formula $t_1 \not\leq t_2$ is unsatisfiable in DLO_Σ^S iff a tableau in which every branch is closed can be constructed starting from the root $\{c : t_1^p, c : \neg t_2^n\}$ and using the rules in $T_\Sigma^S(ext)$, with the restriction that $(\wedge^n), (\vee^n)$ and (f^n) can only be applied if the result is a subexpression of t_2.

5 Conclusions

We formally defined a class of many sorted bridging functions between bounded distributive lattices, showed that the Priestley representation theorem can be extended in a natural way to encompass such operators, and then analyzed some fragments of the universal theory of distributive lattices with many sorted bridging operators. In particular, we showed that a structure-preserving translation to clause form for uniform word problems for such classes of lattices with operators can be defined also in this case using the same pattern used in [SS99] for join hemimorphisms. Using this translation, the results in [SS99] can be extended in a straightforward way to prove that ordered resolution with selection is a decision procedure for the universal theory of many-sorted distributive lattices with bridging operators. We then proved that hyperresolution can be used for simpler problems such as the problem of checking validity of formulae of type $t_1 \leq t_2$ in DLO^S_Σ and DLO^A_Σ. Based on this we sketched a way of designing tableau calculi.

Bridging functions such as "cardinality" are, in general, not join hemimorphisms, but satisfy the subadditivity condition $f(a \vee b) \geq f(a) \vee f(b)$ or conditional additivity axioms such as $x \wedge y = 0 \Rightarrow f(x \vee y) = f(x) + f(y)$. We would like to extend the results presented here to such more general operators.

Acknowledgements. I thank Harald Ganzinger for inspiring discussions, and for bringing the results on bridging functions in [Ohl01] to my attention, and the referees for their helpful comments.

References

[AB75] A.R. Anderson and N.D. Belnap. *Entailment – The Logic of Relevance and Necessity.* Princeton University Press, 1975.

[BG01] L. Bachmair and H. Ganzinger. Resolution theorem proving. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning.* Elsevier, 2001.

[Gol89] R. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44(3):153–301, 1989.

[HS99] U. Hustadt and R. A. Schmidt. On the relation of resolution and tableaux proof systems for description logics. In T. Dean, editor, *Proceedings of IJ-CAI'99*, volume 1, pages 110–115. Morgan Kaufmann, 1999.

[HS00] U. Hustadt and R. A. Schmidt. Using resolution for testing modal satisfiability and building models. In I. P. Gent, H. van Maaren, and T. Walsh, editors, *Proceedings of SAT 2000*, volume 63 of *Frontiers in Artificial Intelligence and Applications.* IOS Press, Amsterdam, 2000.

[JT52] B. Jónsson and A. Tarski. Boolean algebras with operators, Part I&II. *American Journal of Mathematics*, 73&74:891–939&127–162, 1951&1952.

[Ohl01] H.J. Ohlbach. Set description languages and reasoning about numerical features of sets. Technical Report PMS-FB-2001-1, Institut f. Informatik, LMU, München, April 2001.

[Ono93] H. Ono. Semantics for substructural logics. In P. Schroeder-Heister and K. Došen, editors, *Substructural Logics*, pages 259–291. Oxford University Press, 1993.

[Pri70] H.A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.

[SS99] V. Sofronie-Stokkermans. On the universal theory of varieties of distributive lattices with operators: Some decidability and complexity results. In *Proceedings of CADE-16*, LNAI 1632, pages 157–171, Trento, Italy, 1999. Springer.

[SS00] V. Sofronie-Stokkermans. Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics I. *Studia Logica*, 64(1):93–132, 2000.

[SS01] V. Sofronie-Stokkermans. Resolution-based decision procedures for the universal theory of some classes of distributive lattices with operators. Research Report MPI-I-2001-2-005, Max-Planck-Institut für Informatik, Saarbrücken, Germany, September 2001.

[SS02] V. Sofronie-Stokkermans. Representation theorems and the semantics of non-classical logics and applications to automated theorem proving. In M. Fitting and E. Orlowska, editors, *Theory and Applications of Multiple-Valued Logic* Springer-Verlag series Studies in Fuzziness and Soft Computing, to appear, 2002.